

Deformed cohomologies of symmetry pseudo-groups and coverings of differential equations

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Abstract. We discuss a relation between deformed cohomologies of symmetry pseudo-groups and coverings of differential equations. Examples include the potential Khokhlov–Zabolotskaya equation and the Boyer–Finley equation.

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1. Introduction

Deformed (or exotic) cohomologies were introduced in works of S.P. Novikov, [42, 43, 44], as a tool in an analogue of the Morse theory for smooth multi-valued functions. Then they were applied to different problems of symplectic geometry and algebraic topology, [1, 2, 28, 29, 46]. The objective of the present paper is to establish a relation between the deformed cohomologies of symmetry pseudo-groups of partial differential equations and their coverings.

Coverings (or Wahlquist–Estabrook prolongation structures, [52], or zero-curvature representations, [55], or integrable extensions, [4], etc.) are of great importance in geometry of PDEs. The theory of coverings is a natural framework for dealing with nonlocal symmetries and nonlocal conservation laws, inverse scattering constructions for soliton equations, Bäcklund transformations, recursion operators, and deformations of nonlinear PDEs, [16, 17, 18]. A number of techniques has been devised to handle the problem of recognizing whether a given differential equation has a covering, [52, 40, 41, 10, 54, 9, 50, 14, 25, 47, 26, 27]. In [22], examples of coverings of PDEs with three independent variables were found by means of Élie Cartan’s method of equivalence, [5, 6, 7, 12, 15, 45]. This idea was developed in [32, 33, 35]. In [34] we propose an approach to the covering problem based on the technique of contact integrable extensions (CIEs) of the structure equations of the symmetry pseudo-groups, which is a generalization of the definition of integrable extension from [4, §6] for the case of more than two independent variables. Then in [36, 37, 38, 39] the method of CIEs was applied to finding of coverings, Bäcklund transformations and recursion operators for a number of PDEs.

From the definition of deformed cohomology it follows that each non-trivial deformed 2-cocycle of the symmetry pseudo-group of a PDE provides an integrable extension of this pseudo-group. Then a covering for the PDE may be obtained via integration of the extension equation in accordance with Cartan's theorem.

In this paper we consider two equations: the potential Khokhlov–Zabolotskaya equation (or Lin–Reissner–Tsien equation), [23, 53],

$$u_{yy} = u_{tx} + u_x u_{xx}, \quad (1)$$

and the Boyer–Finley equation, [3],

$$u_{tx} = e^{u_y} u_{yy}. \quad (2)$$

We show that symmetry pseudo-groups of both equations have non-trivial deformed second cohomologies. The integrable extensions that correspond to cocycles from these cohomology groups define known coverings of equations (1) and (2).

2. Preliminaries

2.1. Coverings of PDEs

All considerations in this paper are local. The presentation in this subsection closely follows to [19, 20]. Let $\pi: \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$, $\pi: (x^1, \dots, x^n, u^1, \dots, u^m) \mapsto (x^1, \dots, x^n)$ be a trivial bundle, and $J^\infty(\pi)$ be the bundle of its jets of the infinite order. The local coordinates on $J^\infty(\pi)$ are $(x^i, u^\alpha, u_I^\alpha)$, where $I = (i_1, \dots, i_n)$ is a multi-index, and for every local section $f: \mathbb{R}^n \rightarrow \mathbb{R}^n \times \mathbb{R}^m$ of π the corresponding infinite jet $j_\infty(f)$ is a section $j_\infty(f): \mathbb{R}^n \rightarrow J^\infty(\pi)$ such that $u_I^\alpha(j_\infty(f)) = \frac{\partial^{\#I} f^\alpha}{\partial x^I} = \frac{\partial^{i_1+\dots+i_n} f^\alpha}{(\partial x^1)^{i_1} \dots (\partial x^n)^{i_n}}$. We put $u^\alpha = u_{(0,\dots,0)}^\alpha$. Also, in the case of $n = 3$, $m = 1$ we denote $x^1 = t$, $x^2 = x$, $x^3 = y$, and $u_{(i,j,k)}^1 = u_{t\dots tx\dots xy\dots y}$ with i times t , j times x , and k times y .

The vector fields

$$D_{x^k} = \frac{\partial}{\partial x^k} + \sum_{\#I \geq 0} \sum_{\alpha=1}^m u_{I+1_k}^\alpha \frac{\partial}{\partial u_I^\alpha}, \quad k \in \{1, \dots, n\},$$

$(i_1, \dots, i_k, \dots, i_n) + 1_k = (i_1, \dots, i_k + 1, \dots, i_n)$, are called *total derivatives*. They commute everywhere on $J^\infty(\pi)$: $[D_{x^i}, D_{x^j}] = 0$.

A system of PDEs $F_r(x^i, u_I^\alpha) = 0$, $\#I \leq s$, $r \in \{1, \dots, R\}$, of the order $s \geq 1$ with $R \geq 1$ defines the submanifold $\mathcal{E} = \{(x^i, u_I^\alpha) \in J^\infty(\pi) \mid D_K(F_r(x^i, u_I^\alpha)) = 0, \#K \geq 0\}$ in $J^\infty(\pi)$.

Denote $\mathcal{W} = \mathbb{R}^\infty$ with coordinates w^s , $s \in \mathbb{N} \cup \{0\}$. Locally, an (infinite-dimensional) *differential covering* of \mathcal{E} is a trivial bundle $\tau: J^\infty(\pi) \times \mathcal{W} \rightarrow J^\infty(\pi)$ equipped with the *extended total derivatives*

$$\tilde{D}_{x^k} = D_{x^k} + \sum_{s=0}^{\infty} T_k^s(x^i, u_I^\alpha, w^j) \frac{\partial}{\partial w^s} \quad (3)$$

such that $[\tilde{D}_{x^i}, \tilde{D}_{x^j}] = 0$ for all $i \neq j$ whenever $(x^i, u_I^\alpha) \in \mathcal{E}$. We define the partial derivatives of w^s by $w_{x^k}^s = \tilde{D}_{x^k}(w^s)$. This yields the system of *covering equations*

$$w_{x^k}^s = T_k^s(x^i, u_I^\alpha, w^j).$$

This over-determined system of PDEs is compatible whenever $(x^i, u_I^\alpha) \in \mathcal{E}$.

Dually the covering with extended total derivatives (3) is defined by the integrable ideal of the *Wahlquist–Estabrook forms*

$$dw^s - T_k^s(x^i, u_I^\alpha, w^j) dx^k.$$

2.2. Cartan's structure theory of Lie pseudo-groups

Let M be a manifold of dimension n . A *local diffeomorphism* on M is a diffeomorphism $\Phi: \mathcal{U} \rightarrow \hat{\mathcal{U}}$ of two open subsets of M . A *pseudo-group* \mathfrak{G} on M is a collection of local diffeomorphisms of M , which is closed under composition whenever the latter is defined, contains an identity and is closed under inversion. A *Lie pseudo-group* is a pseudo-group whose diffeomorphisms are local analytic solutions of an involutive system of partial differential equations called *defining system*.

Élie Cartan's approach to Lie pseudo-groups is based on a possibility to characterize transformations from a pseudo-group in terms of a set of invariant differential 1-forms called *Maurer–Cartan (MC) forms*. In a general case, MC forms $\omega^1, \dots, \omega^m$ of an infinite-dimensional Lie pseudo-group \mathfrak{G} are defined on a direct product $M \times \tilde{M} \times G$, where \tilde{M} is the coordinate space of parameters of prolongation, [45, Ch. 12], G is a finite-dimensional Lie group, and $m = \dim M + \dim \tilde{M}$. The forms ω^i are independent and include differentials of coordinates on $M \times \tilde{M}$ only, while their coefficients depend also on coordinates of G . These forms characterize the pseudo-group \mathfrak{G} in the following sense: a local diffeomorphism $\Phi: \mathcal{U} \rightarrow \hat{\mathcal{U}}$ on M belongs to \mathfrak{G} whenever there exists a local diffeomorphism $\Psi: \mathcal{W} \rightarrow \hat{\mathcal{W}}$ on $M \times \tilde{M} \times G$ such that $v \circ \Psi = \Phi \circ v$ for the projection $v: M \times \tilde{M} \times G \rightarrow M$ and the forms ω^j are invariant w.r.t. Ψ , that is,

$$\Psi^*(\omega^i|_{\hat{\mathcal{W}}}) = \omega^i|_{\mathcal{W}}. \quad (4)$$

Expressions for $d\omega^i$ in terms of ω^j give Cartan's *structure equations* of \mathfrak{G} :

$$d\omega^i = A_{\gamma j}^i \pi^\gamma \wedge \omega^j + \frac{1}{2} B_{jk}^i \omega^j \wedge \omega^k, \quad B_{jk}^i = -B_{kj}^i. \quad (5)$$

The forms π^γ , $\gamma \in \{1, \dots, \dim G\}$, are linear combinations of MC forms of the Lie group G and the forms ω^i . The coefficients $A_{\gamma j}^i$ and B_{jk}^i are either constants or functions of a set of invariants $U^\kappa: M \rightarrow \mathbb{R}$, $\kappa \in \{1, \dots, l\}$, $l < \dim M$, of the pseudo-group \mathfrak{G} , so $\Phi^*(U^\kappa|_{\hat{\mathcal{U}}}) = U^\kappa|_{\mathcal{U}}$ for every $\Phi \in \mathfrak{G}$. In the latter case, the differentials of U^κ are invariant 1-forms, so they are linear combinations of the forms ω^j ,

$$dU^\kappa = C_j^\kappa \omega^j, \quad (6)$$

where the coefficients C_j^κ depend on the invariants U^1, \dots, U^l only.

Equations (5) must be compatible in the following sense: we have

$$d(d\omega^i) = 0 = d\left(A_{\gamma j}^i \pi^\gamma \wedge \omega^j + \frac{1}{2} B_{jk}^i \omega^j \wedge \omega^k\right), \quad (7)$$

therefore there must exist expressions

$$d\pi^\gamma = W_{\lambda j}^\gamma \chi^\lambda \wedge \omega^j + X_{\beta\epsilon}^\gamma \pi^\beta \wedge \pi^\epsilon + Y_{\beta j}^\gamma \pi^\beta \wedge \omega^j + Z_{jk}^\gamma \omega^j \wedge \omega^k \quad (8)$$

with some additional 1-forms χ^λ such that the right-hand side of (7) is identically equal to zero after substituting for (5), (6), and (8). Also, from (6) it follows that the right-hand side of the equation

$$d(dU^\kappa) = 0 = d(C_j^\kappa \omega^j) \quad (9)$$

must be identically equal to zero after substituting for (5) and (6).

The forms π^γ are not invariant w.r.t. the pseudo-group \mathfrak{G} . Respectively, the structure equations (5) are not changing when replacing $\pi^\gamma \mapsto \pi^\gamma + z_j^\gamma \omega^j$ for certain parametric coefficients z_j^γ . The dimension $r^{(1)}$ of the linear space of these coefficients satisfies the following inequality

$$r^{(1)} \leq n \dim G - \sum_{k=1}^{n-1} (n-k) s_k, \quad (10)$$

where the *reduced characters* s_k are defined by the formulas

$$\begin{aligned} s_1 &= \max_{u_1 \in \mathbb{R}^n} \text{rank } \mathbb{A}_1(u_1), \\ s_k &= \max_{u_1, \dots, u_k \in \mathbb{R}^n} \text{rank } \mathbb{A}_k(u_1, \dots, u_k) - \sum_{j=1}^{k-1} s_j, \quad k \in \{1, \dots, n-1\}, \\ s_n &= \dim G - \sum_{j=1}^{n-1} s_j, \end{aligned}$$

with the matrices \mathbb{A}_k inductively defined by

$$\mathbb{A}_1(u_1) = (A_{\gamma j}^i u_1^j), \quad \mathbb{A}_l(u_1, \dots, u_l) = \begin{pmatrix} \mathbb{A}_{l-1}(u_1, \dots, u_{l-1}) \\ A_{\gamma j}^i u_l^j \end{pmatrix}, \quad l \in \{2, \dots, n-1\},$$

see [5, §5], [45, Def. 11.4] for the full discussion. The system of forms ω^k is *involutive* when both sides of (10) are equal, [5, §6], [45, Def. 11.7].

Cartan's fundamental theorems, [5, §§16, 22–24], [7], [51, §§16, 19, 20, 25, 26], [49, §§14.1–14.3], state that for a Lie pseudo-group there exists a set of MC forms whose structure equations satisfy the compatibility and involutivity conditions; conversely, if equations (5), (6) meet the compatibility conditions (7), (9) and the involutivity condition, then there exists a collection of 1-forms $\omega^1, \dots, \omega^m$ and functions U^1, \dots, U^l which satisfy (5) and (6). Equations (4) then define local diffeomorphisms from a Lie pseudo-group.

EXAMPLE 1. Suppose \mathcal{E} is a second-order differential equation in one dependent and n independent variables. We consider \mathcal{E} as a submanifold in $J^2(\pi)$ with $\pi: \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$. Let $\text{Cont}(\mathcal{E})$ be the group of contact symmetries for \mathcal{E} . It consists of all the contact transformations on $J^2(\pi)$ mapping \mathcal{E} to itself. The MC forms of $\text{Cont}(\mathcal{E})$ can be computed from the MC forms of the pseudo-group all the contact transformations on $J^2(\pi)$ algorithmically by means of Cartan's method of equivalence, [5, 6, 7, 12, 15, 45], see details and examples in [11, 30, 31].

2.3. Deformed cohomologies

Let \mathfrak{g} be a Lie algebra over \mathbb{R} and $\rho: \mathfrak{g} \rightarrow \text{End}(V)$ be its representation. Let $C^k(\mathfrak{g}, V) = \text{Hom}(\Lambda^k(\mathfrak{g}), V)$, $k \geq 1$, be the space of all k -linear skew-symmetric mappings from \mathfrak{g} to V . Then a differential complex

$$V = C^0(\mathfrak{g}, V) \xrightarrow{d} C^1(\mathfrak{g}, V) \xrightarrow{d} \dots \xrightarrow{d} C^k(\mathfrak{g}, V) \xrightarrow{d} C^{k+1}(\mathfrak{g}, V) \xrightarrow{d} \dots$$

is defined by the formula

$$\begin{aligned} d\theta(X_1, \dots, X_{k+1}) &= \sum_{q=1}^{k+1} (-1)^{q+1} \rho(X_q) (\theta(X_1, \dots, \hat{X}_q, \dots, X_{k+1})) \\ &+ \sum_{1 \leq p < q \leq k+1} (-1)^{p+q} \theta([X_p, X_q], X_1, \dots, \hat{X}_p, \dots, \hat{X}_q, \dots, X_{k+1}). \end{aligned}$$

Cohomologies of the complex $(C^*(\mathfrak{g}, V), d)$ are referred to as *cohomologies of the Lie algebra \mathfrak{g} with coefficients in the representation ρ* . For the trivial representation $\rho_0: \mathfrak{g} \rightarrow \mathbb{R}$, $\rho_0: X \mapsto 0$, cohomologies of the corresponding complex are called *cohomologies with trivial coefficients* and denoted by $H^*(\mathfrak{g})$.

Consider a Lie algebra \mathfrak{g} over \mathbb{R} with non-trivial first cohomology group $H^1(\mathfrak{g})$ and take a closed 1-form ω on \mathfrak{g} . Then for any $\lambda \in \mathbb{R}$ define new *deformed differential* $d_{\lambda\omega}: C^k(\mathfrak{g}, \mathbb{R}) \rightarrow C^{k+1}(\mathfrak{g}, \mathbb{R})$ by the formula

$$d_{\lambda\omega}\theta = d\theta + \lambda\omega \wedge \theta.$$

From $d\omega = 0$ it follows that

$$d_{\lambda\omega}^2 = 0. \tag{11}$$

Cohomologies of the complex

$$C^1(\mathfrak{g}, \mathbb{R}) \xrightarrow{d_{\lambda\omega}} \dots \xrightarrow{d_{\lambda\omega}} C^k(\mathfrak{g}, \mathbb{R}) \xrightarrow{d_{\lambda\omega}} C^{k+1}(\mathfrak{g}, \mathbb{R}) \xrightarrow{d_{\lambda\omega}} \dots$$

are referred to as *deformed* (or *exotic*) *cohomologies* of \mathfrak{g} and denoted by $H_{\lambda\omega}^*(\mathfrak{g})$.

REMARK 1. Cohomologies $H_{\lambda\omega}^*(\mathfrak{g})$ coincide with cohomologies of \mathfrak{g} with coefficients in the one-dimensional representation $\rho_{\lambda\omega}: \mathfrak{g} \rightarrow \mathbb{R}$, $\rho_{\lambda\omega}: X \mapsto \lambda\omega(X)$. In particular, when $\lambda = 0$, cohomologies $H_{\lambda\omega}^*(\mathfrak{g})$ coincide with $H^*(\mathfrak{g})$.

REMARK 2. In general, $d_{\lambda\omega}(\alpha \wedge \beta) \neq d_{\lambda\omega}(\alpha) \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge d_{\lambda\omega}(\beta)$.

EXAMPLE 2. Consider the system

$$d\theta^1 = 0, \quad d\theta^2 = -\theta^1 \wedge \theta^2, \quad d\theta^3 = \theta^1 \wedge \theta^3, \quad d\theta^4 = 2\theta^1 \wedge \theta^4, \quad d\theta^5 = \theta^2 \wedge \theta^3$$

for 1-forms $\theta^1, \dots, \theta^5$. This system is compatible, that is, applying d to both sides of its equations and then substituting for the equations themselves into the right-hand sides gives identities $0 = 0$. Therefore the system defines Lie algebra \mathfrak{h} of vectors $X_1,$

... , X_5 such that $\theta^i(X_j) = \delta_j^i$. The forms θ^i are Maurer–Cartan forms of \mathfrak{h} . Evidently, $H^1(\mathfrak{h}) = \mathbb{R}[\theta^1] = \mathbb{R}\theta^1$. Then direct computations give:

$$H_{\lambda\theta^1}^2(\mathfrak{h}) = \begin{cases} \{0\} & \text{for } \lambda \notin \{-3, -2, -1, 1\}, \\ \mathbb{R}[\theta^3 \wedge \theta^4] & \text{for } \lambda = -3, \\ \mathbb{R}[\theta^1 \wedge \theta^4] & \text{for } \lambda = -2, \\ \mathbb{R}[\theta^1 \wedge \theta^3] \oplus \mathbb{R}[\theta^2 \wedge \theta^4] \oplus \mathbb{R}[\theta^3 \wedge \theta^5] & \text{for } \lambda = -1, \\ \mathbb{R}[\theta^1 \wedge \theta^2] \oplus \mathbb{R}[\theta^2 \wedge \theta^5] & \text{for } \lambda = 1. \end{cases}$$

3. Symmetry pseudo-groups of the potential Khoklov-Zabolotskaya and the Boyer-Finley equations

Using the procedures of Élie Cartan’s method of equivalence we find the Maurer–Cartan forms and their structure equations for the symmetry pseudo-groups of equations (1) and (2), see notation in [31].

3.1. Potential Khoklov-Zabolotskaya equation

The structure equations for the symmetry pseudo-group of equation (1) read

$$\begin{aligned} d\theta_0 &= \eta_1 \wedge \theta_0 + \xi^1 \wedge \theta_1 + \xi^2 \wedge \theta_2 + \xi^3 \wedge \theta_3, \\ d\theta_1 &= (\eta_1 - \eta_2) \wedge \theta_1 - 2\eta_3 \wedge \theta_3 - \theta_0 \wedge (\xi^2 + \sigma_{22}) + \xi^1 \wedge \sigma_{11} + \xi^2 \wedge \sigma_{12} + \xi^3 \wedge \sigma_{13}, \\ d\theta_2 &= \frac{1}{2}(\eta_1 - \eta_2) \wedge \theta_2 + \xi^1 \wedge \sigma_{12} + \xi^2 \wedge \sigma_{22} + \xi^3 \wedge \sigma_{23}, \\ d\theta_3 &= \frac{3}{4}(\eta_1 - \eta_2) \wedge 3\theta_3 - \eta_3 \wedge \theta_2 + \xi^1 \wedge \sigma_{13} + \xi^2 \wedge \sigma_{23} + \xi^3 \wedge \sigma_{12}, \\ d\xi^1 &= \eta_2 \wedge \xi^1, \\ d\xi^2 &= \frac{1}{2}(\eta_1 + \eta_2) \wedge \xi^2 + \eta_3 \wedge \xi^3 - \theta_2 \wedge \xi^1, \\ d\xi^3 &= \frac{1}{4}(\eta_1 + 3\eta_2) \wedge \xi^3 + 2\eta_3 \wedge \xi^1, \\ d\sigma_{11} &= (\eta_1 - 2\eta_2) \wedge \sigma_{11} - 4\eta_3 \wedge \sigma_{13} - (\eta_4 - \theta_2) \wedge \theta_0 + \eta_6 \wedge \xi^2 + \eta_7 \wedge \xi^3 + \eta_8 \wedge \xi^1 \\ &\quad - 5\theta_1 \wedge (\xi^2 + \sigma_{22}) + \theta_2 \wedge \sigma_{12} - 2\theta_3 \wedge \sigma_{23}, \\ d\sigma_{12} &= \frac{1}{2}(\eta_1 - 3\eta_2) \wedge \sigma_{12} - 2\eta_3 \wedge \sigma_{23} + \eta_4 \wedge \xi^2 + \eta_5 \wedge \xi^3 + \eta_6 \wedge \xi^1 - 2\theta_2 \wedge (\xi^2 + \sigma_{22}), \\ d\sigma_{13} &= \frac{1}{4}(3\eta_1 - 7\eta_2) \wedge 3\sigma_{13} - 3\eta_3 \wedge \sigma_{12} + \eta_5 \wedge \xi^2 + \eta_6 \wedge \xi^3 + \eta_7 \wedge \xi^1 - 3\theta_3 \wedge (\xi^2 + \sigma_{22}), \\ d\sigma_{22} &= \eta_4 \wedge \xi^1 - \frac{1}{2}\eta_1 \wedge \xi^2 - \frac{1}{2}\eta_2 \wedge (3\xi^2 + 2\sigma_{22}) - \eta_3 \wedge \xi^3, \\ d\sigma_{23} &= \frac{1}{4}(\eta_1 - 5\eta_2) \wedge \sigma_{23} - \eta_3 \wedge (\xi^2 + \sigma_{22}) + (\eta_4 - \theta_2) \wedge \xi^3 + \eta_5 \wedge \xi^1, \\ d\eta_1 &= \xi^1 \wedge (\xi^2 + \sigma_{22}), \\ d\eta_2 &= -3\xi^1 \wedge (\xi^2 + \sigma_{22}), \\ d\eta_3 &= \frac{1}{4}(\eta_1 - \eta_2) \wedge \eta_3 + \sigma_{23} \wedge \xi^1 - \xi^3 \wedge (\xi^2 + \sigma_{22}), \end{aligned}$$

$$\begin{aligned}
 d\eta_4 &= \eta_9 \wedge \xi^1 + \frac{1}{2}(\eta_2 \wedge (3\theta_2 - 4\eta_4) + \eta_1 \wedge \theta_2) + \xi^2 \wedge \sigma_{22} + \xi^3 \wedge \sigma_{23}, \\
 d\eta_5 &= \eta_9 \wedge \xi^3 + \eta_{10} \wedge \xi^1 + \frac{1}{4}(\eta_1 - 9\eta_2) \wedge \eta_5 - 3\eta_3 \wedge (\eta_4 - \theta_2) + \sigma_{12} \wedge \xi^3 \\
 &\quad + 3\sigma_{23} \wedge (\xi^2 + \sigma_{22}), \\
 d\eta_6 &= \eta_9 \wedge \xi^2 + \eta_{10} \wedge \xi^3 + \eta_{11} \wedge \xi^1 + \frac{1}{2}(\eta_1 - 5\eta_2) \wedge \eta_6 - 4\eta_3 \wedge \eta_5 + \sigma_{12} \wedge (7\xi^2 + 6\sigma_{22}), \\
 d\eta_7 &= \eta_{10} \wedge \xi^2 + \eta_{11} \wedge \xi^3 + \eta_{12} \wedge \xi^1 + \frac{1}{4}(3\eta_1 - 11\eta_2) \wedge \eta_7 - 5\eta_3 \wedge \eta_6 + 3\eta_4 \wedge \theta_3 \\
 &\quad - (\eta_5 - 3\theta_3) \wedge \theta_2 + 3\sigma_{12} \wedge \sigma_{23} + 9\sigma_{13} \wedge (\xi^2 + \sigma_{22}), \\
 d\eta_8 &= (\eta_1 - 3\eta_2) \wedge \eta_8 + 6(\eta_4 \wedge \theta_1 - \eta_3 \wedge \eta_7 + \sigma_{13} \wedge \sigma_{23}) + 2\eta_5 \wedge \theta_3 - 2(\eta_6 - 3\theta_1) \wedge \theta_2 \\
 &\quad - (\eta_9 + \sigma_{12}) \wedge \theta_0 + \eta_{11} \wedge \xi^2 + \eta_{12} \wedge \xi^3 + \eta_{13} \wedge \xi^1 + 12\sigma_{11} \wedge (\xi^2 + \sigma_{22}). \tag{12}
 \end{aligned}$$

We have the following Maurer–Cartan forms

$$\begin{aligned}
 \xi^1 &= \frac{dt}{a}, \\
 \xi^2 &= a \left(\frac{u_{xy}^2 - u_x u_{xxx}^2}{u_{xxx}} dt + u_{xxx} dx + u_{xy} dy \right), \\
 \xi^3 &= 2 \frac{u_{xy}}{u_{xxx}^{1/2}} dt + u_{xxx}^{1/2} dy, \\
 \theta_2 &= a^2 u_{xxx} (du_x - u_{tx} dt - u_{xx} dx - u_{xy} dy), \\
 \theta_3 &= a^3 u_{xxx}^{1/2} (u_{xxx} (du_y - u_{ty} dt - u_{xy} dx - (u_{tx} + u_x u_{xx}) dy) \\
 &\quad - u_{xy} (du_x - u_{tx} dt - u_{xx} dx - u_{xy} dy)), \\
 \eta_1 &= 3 \frac{da}{a} + 2 \frac{du_{xxx}}{u_{xxx}} - u_{xx} dt, \\
 \eta_2 &= -\frac{da}{a} + 3u_{xx} dt, \\
 \eta_3 &= a \left(\frac{du_{xy}}{u_{xxx}^{1/2}} - \frac{u_{xy} du_{xxx}}{u_{xxx}^{3/2}} + \frac{(u_{xx} u_{xy} + u_{xy} u_{xxx}) dt}{u_{xxx}^{1/2}} + u_{xx} u_{xxx}^{1/2} dy \right), \tag{13}
 \end{aligned}$$

where $a \neq 0$ is a parameter. We do not need explicit expressions for the other MC forms of this pseudo-group in what follows.

3.2. The Boyer–Finley equation

For the symmetry pseudo-group of equation (2) we have the following structure equations

$$\begin{aligned}
d\theta_0 &= \theta_0 \wedge (\theta_3 - \sigma_{33}) + \xi^1 \wedge \theta_1 + \xi^2 \wedge \theta_2 + \xi^3 \wedge \theta_3, \\
d\theta_1 &= \eta_1 \wedge \theta_1 + \xi^1 \wedge \sigma_{11} + \xi^2 \wedge \sigma_{33} + \xi^3 \wedge \sigma_{13}, \\
d\theta_2 &= \theta_2 \wedge (\eta_1 + \theta_3 + \xi^3) + \xi^1 \wedge \sigma_{33} + \xi^2 \wedge \sigma_{22} + \xi^3 \wedge \sigma_{23}, \\
d\theta_3 &= \xi^1 \wedge \sigma_{13} + \xi^2 \wedge \sigma_{23} + (\theta_3 + \sigma_{33}) \wedge \xi^3, \\
d\xi^1 &= (\sigma_{33} - \theta_3 - \eta_1) \wedge \xi^1, \\
d\xi^2 &= (\eta_1 + \sigma_{33} + \xi^3) \wedge \xi^2, \\
d\xi^3 &= (\sigma_{33} - \theta_3) \wedge \xi^3, \\
d\sigma_{11} &= (2\eta_1 + \theta_3 - \sigma_{33}) \wedge \sigma_{11} + \eta_2 \wedge \xi^3 + \eta_3 \wedge \xi^1 - \sigma_{13} \wedge \xi^2 + \theta_1 \wedge (\xi^2 + \sigma_{13}), \\
d\sigma_{13} &= (\eta_1 + \theta_3 - \sigma_{33}) \wedge \sigma_{13} + \eta_2 \wedge \xi^1 + (\theta_3 + 2\sigma_{33} - \xi^3) \wedge \xi^2, \\
d\sigma_{22} &= \sigma_{22} \wedge (2\eta_1 + \theta_3 + 2\xi^3 + \sigma_{33}) + \eta_4 \wedge \xi^3 + \eta_5 \wedge \xi^2 + \theta_2 \wedge (\xi^1 + \sigma_{23}) - \sigma_{23} \wedge \xi^1, \\
d\sigma_{23} &= \sigma_{23} \wedge (\eta_1 + \xi^3 + \sigma_{33}) + \eta_4 \wedge \xi^2 + (\theta_3 + 2\sigma_{33} - \xi^3) \wedge \xi^1, \\
d\sigma_{33} &= \xi^1 \wedge \sigma_{13} + \xi^2 \wedge \sigma_{23} + \xi^3 \wedge (\sigma_{33} - \theta_3), \\
d\eta_1 &= (\sigma_{13} + \xi^2) \wedge \xi^1, \\
d\eta_2 &= \eta_6 \wedge \xi^1 + 2(\eta_1 + \theta_3 - \sigma_{33}) \wedge \eta_2, \\
d\eta_3 &= \eta_6 \wedge \xi^3 + \eta_7 \wedge \xi^1 + (3\eta_1 + 2(\theta_3 - \sigma_{33})) \wedge \eta_3 - \eta_2 \wedge (\theta_1 + \xi^2) - 3\sigma_{11} \wedge (\xi^2 + \sigma_{13}), \\
d\eta_4 &= \eta_8 \wedge \xi^2 - 2(\eta_1 + \xi^3 + \sigma_{33}) \wedge \eta_4, \\
d\eta_5 &= \eta_8 \wedge \xi^3 + \eta_9 \wedge \xi^2 - (3\eta_1 + \theta_3 + 3\xi^3 + 2\sigma_{33}) \wedge \eta_5 - \eta_4 \wedge (\theta_2 + \xi^1) \\
&\quad - 3\sigma_{22} \wedge (\xi^1 + \sigma_{23}).
\end{aligned} \tag{14}$$

In what follows we need explicit expressions for the following MC forms only:

$$\begin{aligned}
\theta_1 &= \frac{u_{yy}}{a} (du_x - e^{u_y} u_{yy} dt - u_{xx} dx - u_{xy} dy), \\
\theta_3 &= du_y - u_{ty} dt - u_{xy} dx - u_{yy} dy, \\
\xi^1 &= a dt, \\
\xi^2 &= \frac{e^{u_y} u_{yy}^2}{a} dx, \\
\xi^3 &= u_{yy} dy, \\
\sigma_{33} &= \frac{du_{yy}}{u_{yy}} + \theta_3, \\
\eta_1 &= -\frac{da}{a} + \frac{u_{ty}}{a} \xi^1 + \sigma_{33} - \theta_3.
\end{aligned} \tag{15}$$

4. Deformed cohomologies and coverings

Now we consider infinite-dimensional Lie algebras \mathfrak{g}_1 and \mathfrak{g}_2 defined by normal prolongations, [5, 7, 51, 49], of systems (12) and (14), respectively, and study their second deformed cohomologies. In both cases they appear to be non-zero. Non-trivial 2-cocycles define integrable extensions, [4, 34], of (12) and (14). Solutions to the integrable extensions coincide with the Wahlquist–Estabrook forms of known coverings of equations (1) and (2).

4.1. The potential Khoklov-Zabolotskaya equation

It is simple to show that $H^1(\mathfrak{g}_1)$ is generated by 1-form

$$\zeta = 3\eta_1 + \eta_2.$$

Denote by I the exterior ideal generated by 1-forms θ_i , $0 \leq i \leq 3$, ξ^j , $1 \leq j \leq 3$, σ_{11} , σ_{12} , σ_{13} , σ_{22} , σ_{23} , η_k , $1 \leq k \leq 8$, that is, the left hand sides of equations (12) contain differentials of the generators of I . Then direct computations show that

$$\dim (H_{\lambda\zeta}^2(\mathfrak{g}_1) \cap I) = \begin{cases} 0, & \lambda \neq -\frac{1}{4}, \\ 1, & \lambda = -\frac{1}{4}, \end{cases}$$

and

$$H_{-\frac{1}{4}\zeta}^2(\mathfrak{g}_1) \cap I = \mathbb{R}[\Omega],$$

where

$$\Omega = \frac{1}{4}(\eta_1 - \eta_2 - 4\eta_3) \wedge (\xi^1 + \xi^2 + \xi^3) + \theta_2 \wedge (\xi^1 + \xi^3) + \theta_3 \wedge \xi^1.$$

We have a conjecture that

$$\dim H_{\lambda\zeta}^2(\mathfrak{g}_1) = \begin{cases} 0, & \lambda \neq -\frac{1}{4}, \\ 1, & \lambda = -\frac{1}{4}. \end{cases}$$

From (11) it follows that equation

$$d\omega - \frac{1}{4}\zeta \wedge \omega = \Omega$$

is compatible with system (12), that is, defines an integrable extension, [4, 34], of (12). Therefore Lie's third inverse fundamental theorem in Cartan's form, [5, 7, 51, 49], ensures existence of a solution $\omega \notin I$ to the equation

$$\begin{aligned} d\omega &= \frac{1}{4}(3\eta_1 + \eta_2) \wedge \omega + \\ &\quad \frac{1}{4}(\eta_1 - \eta_2 - 4\eta_3) \wedge (\xi^1 + \xi^2 + \xi^3) + \theta_2 \wedge (\xi^1 + \xi^3) + \theta_3 \wedge \xi^1. \end{aligned}$$

Since forms (13) are known, we can find ω explicitly:

$$\omega = a^2 u_{xxx}^{3/2} \left(dq - \left(\frac{1}{3} q_x^3 - u_x q_x - u_y \right) dt - q_x dx - \left(\frac{1}{2} q_x^2 - u_x \right) dy \right).$$

In this expression q is a new variable (an “integration constant”), while the new parameter q_x can be expressed in terms of the free parameter a of the MC forms of the symmetry pseudo-group from the relation

$$a = \frac{u_{xxx}^{1/2}}{u_{xxx} q_x - u_{xy}}.$$

The condition $\omega = 0$ gives the covering system

$$\begin{cases} q_t &= \frac{1}{3} q_x^3 - u_x q_x - u_y, \\ q_y &= \frac{1}{2} q_x^2 - u_x \end{cases}$$

for equation (1). This covering was found in [22] and then in [21, 13].

4.2. The Boyer-Finley equation

For the symmetry pseudo-group of the Boyer-Finley equation we have $H^1(\mathfrak{g}_2) = \mathbb{R}\zeta$ with

$$\zeta = \sigma_{33} - \theta_3$$

and

$$\dim (H_{\lambda\zeta}^2(\mathfrak{g}_2) \cap I) = \begin{cases} 0, & \lambda \neq -1, \\ 2, & \lambda = -1, \end{cases}$$

where I is the exterior ideal generated by θ_i , $0 \leq i \leq 3$, ξ^j , $1 \leq j \leq 3$, σ_{11} , σ_{13} , σ_{22} , σ_{23} , σ_{33} , η_k , $1 \leq k \leq 5$, while

$$H_{-\zeta}^2(\mathfrak{g}_2) = \mathbb{R}[\Omega_1] \oplus \mathbb{R}[\Omega_2]$$

with

$$\Omega_1 = \eta_1 \wedge (\xi^1 + \xi^2 + \xi^3) - (\theta_1 + \xi^2) \wedge \xi^1 + (\theta_3 + \xi^3) \wedge \xi^2$$

and

$$\Omega_2 = (\sigma_{33} - \theta_3) \wedge \xi^3.$$

We have a conjecture that

$$\dim H_{\lambda\zeta}^2(\mathfrak{g}_2) = \begin{cases} 0, & \lambda \neq -1, \\ 2, & \lambda = -1. \end{cases}$$

The integrable extension which corresponds to Ω_1

$$d\omega = (\sigma_{33} - \theta_3) \wedge \omega + \eta_1 \wedge (\xi^1 + \xi^2 + \xi^3) - (\theta_1 + \xi^2) \wedge \xi^1 + (\theta_3 + \xi^3) \wedge \xi^2$$

has the following solution

$$\omega = u_{yy} (dq - (u_t + e^{qy}) dt + e^{u_y - qy} dx - q_y dy)$$

with the relation $a = u_{yy} e^{q_y}$ between the free parameter a and the new parameter q_x . The corresponding covering

$$\begin{cases} q_t &= u_t + e^{q_y}, \\ q_x &= -e^{u_y - q_y} \end{cases}$$

was found independently in [54, 48, 24].

The solution to equation $d\omega = (\sigma_{33} - \theta_3) \wedge \omega + \Omega_2$ reads $\omega = u_{yy} (dq + \ln u_{yy} dy)$ and is not interesting.

5. Conclusion

We have shown that for some PDEs their coverings arise quite naturally from the second deformed cohomologies of their symmetry pseudo-groups. It would be interesting to find out whether coverings of other PDEs can be derived using this construction. The further research will include clarification of relations between this technique and the other approaches to finding differential coverings. This also leads to the question of other applications of deformed cohomologies of infinite-dimensional Lie algebras as well as to the problem of improving the methods of their study.

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